TUTORIAL NOTES FOR MATH4220

JUNHAO ZHANG

1. THE MEAN VALUE THEOREMS AND THE MAXIMUM PRINCIPLES

Recall the mean value theorems and the maximum principles.

Theorem 1 (Mean value theorems). *Let $u \in C^2(\Omega)$ satisfy $\Delta u = 0$ $(\geq 0,\ \leq 0)$ in Ω. Then we have*

$$u(x) = (\leq, \geq)\frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)dS_y, \quad \forall B_r(x) \subset\subset \Omega,$$

or

$$u(x) = (\leq, \geq)\frac{n}{\omega_n r^n}\int_{B_r(x)} u(y)dy, \quad \forall B_r(x) \subset\subset \Omega,$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$.

Theorem 2 (Strong maximum principle). *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ with $\Delta u \geq 0$ (≤ 0) in Ω, and suppose there exists a point $y \in \Omega$ for which $u(y) = \sup_\Omega u$ $(\inf_\Omega u)$. Then u is constant. Consequently a harmonic function cannot assume an interior maximum or minimum value unless it is constant.*

Theorem 3 (Weak maximum principle). *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ with $\Delta u \geq 0$ (≤ 0) in Ω. Then, provided Ω is bounded,*

$$\sup_\Omega u = \sup_{\partial\Omega} u \ (\inf_\Omega u = \inf_{\partial\Omega} u).$$

Consequently, for harmonic u

$$\inf_{\partial\Omega} u \leq u(x) \leq \sup_{\partial\Omega} u, \quad \forall x \in \Omega.$$

Let us show some results given by the mean value theorems and the maximum principles.

Example 4 (Bernstein). Suppose u is harmonic in B_1. Then there holds

$$\sup_{B_{\frac{1}{2}}} |Du| \leq c \sup_{\partial B_1} |u|,$$

where $c = c(n)$ is a positive constant. In particular for any $\alpha \in [0, 1]$ there holds

$$|u(x) - u(y)| \leq c|x - y|^\alpha \sup_{\partial B_1} |u|, \quad \forall x, y \in B_{\frac{1}{2}},$$

where $c = c(n, \alpha)$ is a positive constant.

Proof. Direct computation shows that

$$\Delta(|Du|^2) = 2\sum_{i,j=1}^n (D_{ij}u)^2 + 2\sum_{i=1}^n D_i u D_i(\Delta u) = 2\sum_{i,j=1}^n (D_{ij}u)^2,$$

moreover,

$$\Delta(\varphi|Du|^2) = (\Delta\varphi)|Du|^2 + 4\sum_{i,j=1}^{n} D_i\varphi D_j u D_{ij}u + 2\varphi\sum_{i,j=1}^{n}(D_{ij}u)^2, \quad \forall\varphi\in C_0^1(B_1).$$

By taking $\varphi=\eta^2$ for some $\eta\in C_0^1(B_1)$ with $\eta\equiv 1$ in $B_{\frac{1}{2}}$, we obtain by the Hölder's inequality,

$$\begin{aligned}\Delta(\eta^2|Du|^2) &= 2\eta\Delta\eta|Du|^2 + 2|D\eta|^2|Du|^2 + 8\eta\sum_{i,j=1}^{n} D_i\eta D_j u D_{ij}u + 2\eta^2\sum_{i,j=1}^{n}(D_{ij}u)^2\\ &\geq (2\eta\Delta\eta - 6|D\eta|^2)|Du|^2 \geq -C|Du|^2,\end{aligned}$$

where C is a positive constant depending only on η. Moreover, since

$$\Delta(u^2) = 2|Du|^2 + 2u\Delta u = 2|Du|^2,$$

by choosing α large enough we get

$$\Delta(\eta^2|Du|^2 + \alpha u^2) \geq 0,$$

then by the maximum principle, we have

$$\sup_{B_1}(\eta^2|Du|^2+\alpha u^2) \leq \sup_{\partial B_1}(\eta^2|Du|^2+\alpha u^2),$$

which implies

$$\sup_{B_{\frac{1}{2}}}|Du| \leq c\sup_{\partial B_1}|u|,$$

where $c=c(n)$ is a positive constant. Therefore we have

$$|u(x)-u(y)| \leq c|x-y|\sup_{\partial B_1}|u|, \quad \forall x,y\in B_{\frac{1}{2}}.$$

□

Example 5 (Harnack inequality)**.** Suppose u is a non-negative harmonic function in $B_{\frac{1}{2}}$. Then there holds

$$\sup_{B_{\frac{1}{2}}}|D\log u| \leq C,$$

where $C=C(n)$ is a positive constant. In particular, there holds

$$u(x) \leq Cu(y), \quad \forall x,y\in B_{\frac{1}{2}},$$

where $C=C(n)$ is a positive constant.

Proof. It suffices to consider $u>0$ in B_1. Set $v=\log u$. Then direct computation shows

$$\Delta v = -|Dv|^2,$$

and set $w=|Dv|^2$, we get

$$\Delta w + 2\sum_{i=1}^{n} D_i v D_i w = 2\sum_{i,j=1}^{n}(D_{ij}v)^2,$$

moreover, take $\varphi \in C_0^1(B_1)$ with $\frac{|D\varphi|^2}{\varphi}$ bounded in B_1, by Hölder's inequality,

$$\begin{aligned}
&\Delta(\varphi w) + 2\sum_{i=1}^n D_i v D_i(\varphi w) \\
&= 2\varphi \sum_{i,j=1}^n (D_{ij}v)^2 + 4\sum_{i,j=1}^n D_i\varphi D_j v D_{ij} v + 2w\sum_{i=1}^n D_i\varphi D_i v + (\Delta\varphi)w \\
&\geq \varphi \sum_{i,j=1}^n (D_{ij}v)^2 - 2|D\varphi||Dv|^3 - \left(|\Delta\varphi| + C\frac{|D\varphi|^2}{\varphi}\right)|Dv|.
\end{aligned}$$

Choose $\varphi = \eta^4$ for some $\eta \in C_0^1(B_1)$. Hence for such fixed η we obtain

$$\begin{aligned}
&\Delta(\eta^4 w) + 2\sum_{i=1}^n D_i v D_i(\eta^4 w) \\
&\geq \frac{1}{n}\eta^4|Dv|^4 - C\eta^3|D\eta||Dv|^3 - 4\eta^2(\eta\Delta\eta + C|D\eta|^2)|Dv|^2 \\
&\geq \frac{1}{n}\eta^4|Dv|^4 - C\eta^3|Dv|^3 - C\eta^2|Dv|^2,
\end{aligned}$$

where C is a positive constant depending only on n and η. Hence we get by Hölder's inequality

$$\Delta(\eta^4 w) + 2\sum_{i=1}^n D_i v D_i(\eta^4 w) \geq \frac{1}{n}\eta^4 w^2 - C,$$

where C is a positive constant depending only on n and η.

Suppose $\eta^4 w$ attains its maximum at $x_0 \in B_1$. Then $D(\eta^4 w) = 0$ and $\Delta(\eta^4 w) \leq 0$ at x_0. Hence there holds

$$\eta^4 w^2(x_0) \leq C.$$

If $w(x_0) \geq 1$, then $\eta^4 w(x_0) \leq C$. Otherwise $\eta^4 w(x_0) \leq \eta^4(x_0)$. In both cases we conclude

$$\eta^4 w(x) \leq C, \quad \forall x \in B_1.$$

For any $x, y \in B_{\frac{1}{2}}$, by simple integration we obtain

$$\log\frac{u(x)}{u(y)} \leq |x-y| \int_0^1 |D\log u(tx + (1-t)y)|dt \leq C|x-y|,$$

therefore

$$u(x) \leq Cu(y).$$

□

Example 6 (Hölder continuity)**.** Suppose u is a harmonic function in B_1 with $u = \varphi$ on ∂B_1. If $\varphi \in C^\alpha(\partial B_1)$ for some $\alpha \in (0,1)$, then $u \in C^{\frac{\alpha}{2}}(\bar{B}_1)$. Moreover, there holds

$$\|u\|_{C^{\frac{\alpha}{2}}(\bar{B}_1)} \leq C\|\varphi\|_{C^\alpha(\partial B_1)},$$

where $C = C(n, \alpha)$ is a positive constant.

Proof. First the maximum principle implies that

$$\inf_{\partial B_1}\varphi \leq u(x) \leq \sup_{\partial B_1}\varphi, \quad \forall x \in B_1.$$

moreover, we claim that

$$\sup_{x\in B_1}\frac{|u(x)-u(x_0)|}{|x-x_0|^{\frac{\alpha}{2}}}\leq 2^{\frac{\alpha}{2}}\sup_{x\in\partial B_1}\frac{|\varphi(x)-\varphi(x_0)|}{|x-x_0|^{\alpha}},\quad \forall x_0\in\partial B_1.$$

Indeed, without loss of generality, we assume $B_1 = B_1((1,0,\cdots,0))$, $x_0 = 0$ and $\varphi(0) = 0$. Define $K = \sup\limits_{x\in\partial B_1}\frac{|\varphi(x)|}{|x|^{\alpha}}$. Since $|x|^2 = 2x_1$ for $x \in \partial B_1$. Therefore for $x \in \partial B_1$ there holds

$$\varphi(x)\leq K|x|^{\alpha}\leq 2^{\frac{\alpha}{2}}Kx_1^{\frac{\alpha}{2}},$$

define $v(x) = 2^{\frac{\alpha}{2}}Kx_1^{\frac{\alpha}{2}}$ in B_1. Then we have

$$\Delta v(x)=2^{\frac{\alpha}{2}}K\cdot\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)x_1^{\frac{\alpha}{2}-2}<0,\quad\forall x\in B_1,$$

therefore

$$u(x)\leq v(x)=2^{\frac{\alpha}{2}}Kx_1^{\frac{\alpha}{2}}\leq 2^{\frac{\alpha}{2}}K|x|^{\frac{\alpha}{2}},\quad\forall x\in B_1,$$

considering $-u$ similarly, we get

$$|u(x)|\leq 2^{\frac{\alpha}{2}}K|x|^{\frac{\alpha}{2}},\forall x\in B_1.$$

The result follows from the above estimates. Indeed, for any $x, y \in B_1$, set $d_x = dist(x, \partial B_1)$ and $d_y = dist(y, \partial B_1)$. Suppose $d_y \leq d_x$. Take $x_0, y_0 \in \partial B_1$ such that $|x - x_0| = d_x$ and $|y - y_0| = d_y$.

If $|x-y| \leq \frac{d_x}{2}$. Then $y \in \bar{B}_{\frac{d_x}{2}}(x) \subset B_{d_x}(x) \subset B_1$. Therefore

$$d_x^{\frac{\alpha}{2}}\frac{|u(x)-u(y)|}{|x-y|^{\frac{\alpha}{2}}}\leq C|u-u(x_0)|_{L^\infty(B_{d_x}(x))}\leq Cd_x^{\frac{\alpha}{2}}\|\varphi\|_{C^\alpha(\partial B_1)}.$$

Hence we obtain

$$|u(x)-u(y)|\leq C|x-y|^{\frac{\alpha}{2}}\|\varphi\|_{C^\alpha(\partial B_1)}.$$

If $d_y \leq d_x \leq 2|x-y|$. Then

$$\begin{aligned}|u(x)-u(y)|&\leq|u(x)-u(x_0)|+|u(x_0)-u(y_0)|+|u(y_0)-u(y)|\\&\leq C(d_x^{\frac{\alpha}{2}}+|x_0-y_0|^{\frac{\alpha}{2}}+d_y^{\frac{\alpha}{2}})\|\varphi\|_{C^\alpha(\partial B_1)}\\&\leq C|x-y|^{\frac{\alpha}{2}}\|\varphi\|_{C^\alpha(\partial B_1)}.\end{aligned}$$

□

A Supplementary Problem

In a bounded region $\Omega \subset \mathbb{R}^n$, if u satisfies

$$\begin{aligned}\Delta u=&f\quad\text{in }\Omega,\\u=&g\quad\text{on }\partial\Omega.\end{aligned}$$

show that

$$|u(x)|\leq\sup_{x\in\partial\Omega}|g(x)|+C\sup_{x\in\Omega}|f(x)|,\quad\forall x\in\bar{\Omega},$$

where $C = C(\Omega)$ is a positive constant.

For more materials, please refer to [1, 2, 3, 4].

References

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Email address: jhzhang@math.cuhk.edu.hk